MUTUAL INTERSECTION FOR ROUGH DIFFERENTIAL SYSTEMS DRIVEN BY FRACTIONAL BROWNIAN MOTIONS

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ABSTRACT. Let X^H and X^K be solutions to two stochastic differential equations driven by independent fractional Brownian motions with Hurst parameters H and K, respectively. We study when X^H and X^K intersects with each other over finite time interval.

1. INTRODUCTION

Random dynamical systems are well established modeling tools for a variety of natural phenomena ranging from physics (fundamental and phenomenological) to chemistry and more recently to biology, economics, engineering sciences and mathematical finance. In many interesting models the lack of any regularity of the external inputs of the differential equation as functions of time is a technical difficulty that hampers their mathematical analysis. The theory of rough paths has been initially developed by T. Lyons [14] in the 1990's to provide a framework to analyze a large class of driven differential equations and the precise relations between the driving signal and the output (that is the state, as function of time, of the controlled system).

Rough paths theory provides a nice framework to study differential equations driven by Gaussian processes (see [8]). In particular, using rough paths theory, we may define solutions of stochastic differential equations driven by a fractional Brownian motion. Consider

$$X_t = x + \int_0^t V_0(X_s)ds + \sum_{i=1}^{d} \int_0^t V_i(X_s)dB_s^i, \tag{1}$$

where $x \in \mathbb{R}^n$, $V_0, V_1, \cdots, V_d$ are bounded smooth vector fields on $\mathbb{R}^n$ and $\{B_t, t \geq 0\}$ is a d-dimensional fractional Brownian motion with Hurst parameter $H \in (1/4, 1)$. Existence and uniqueness of solutions to the above equation can be found, for example, in [15]. In particular, when $H = 1/2$, this notion of solution coincides with the solution of the corresponding Stratonovitch stochastic differential equation. It is also clear now (cf. [1, 3, 5, 10, 4]) that under Hörmander's condition the law of the solution X_t has a smooth density $p_t(x, y)$ with respect to the Lebesgue measure on $\mathbb{R}^n$.

In the present work, we are interested in the mutual intersection of two independent solutions to equation (1). More precisely, Suppose we have two mutually independent fractional Brownian motions $B = (B^1, ..., B^d)$ and $\tilde{B} = (\tilde{B}^1, ..., \tilde{B}^d)$ from the same probability space $(\Omega, \mathscr{F}, \mathbb{P})$, with Hurst parameters H and K, respectively. We assume that both H and K are greater than $1/4$. Note here that we give ourselves the flexibility that the two fractional Brownian motions have different Hurst parameters. Let X^H be the solution to equation (1) driven by B. Assume that X^K is the solution to an equation of the same type as (1), but with a different starting point $\tilde{x}$ and possibly another set of vector fields $\tilde{V}_i : i = 0, ..., d$, driven by the second fractional Brownian motion $\tilde{B}$. Clearly X^H and X^K are independent.

We are interested in when these two solutions intersect with each other over the time interval $[0, 1]$.

The question of mutual intersection as above are usually discussed in the setting of random fields. Standard strategy in solving this problem is to consider the random field of two parameters $Y(s,t) = (X_s^H, X_t^K)$ and translate the question of mutual intersection to the question of hitting probability

$$\text{When do we have } \mathbb{P}\{\, Y \text{ hits } D \ \text{ on } [0,1]^2\,\} > 0\ ?$$

Here $D = \{(x,x) : x \in \mathbb{R}^n\}$ is the diagonal.

The problem about hitting probabilities is important in potential theory of stochastic processes and random fields. Usually, to solve a hitting probability problem, sophisticated computations are expected. We refer to [6, 16] and references therein for details.

In this work, we propose a simple approach to the problem, employing current development in the study of equation (1). The main idea is described as follows. Since X^H and X^K are independent, we can freeze X^H by conditioning. For a single sample path $X^H[0,1](\omega) = \{X_t^H(\omega) : 0 \le t \le 1\}$, one knows its Hausdorff dimension (as a subset of $\mathbb{R}^n$) explicitly in terms of H (see Theorem 3.2 below). On the other hand, it is also known that for any bounded Borel set $E \subset \mathbb{R}^d$ the probability

$$\mathbb{P}\,(X_t \text{ hits } E \text{ for } t \in [a,b])$$

can be characterized by the α-dimensional Newtonian capacity of E for $\alpha = n - 1/K$ (see Theorem 3.1 below). Given the relation between Hausdorff dimension and Capacity dimension, one should be able to draw some information on whether X^K hits a particular sample path $E = X^H[0,1](\omega)$ of X^H. The question whether X^K hits X^H is then answered by undoing the conditioning.

Throughout our discussion, we assume that the vector fields V_i (and $\tilde{V}_i$, respectively) in equation (1) for X^H (and X^K, respectively) are C^∞-bounded and satisfy the following uniform ellipticity condition.

Hypothesis 1.1 (Uniform Ellipticity)**.** *The vector fields $V_1, \ldots, V_d$ are said to form an uniform elliptic system if*

$$v^* V(x) V^*(x) v \ge \lambda |v|^2, \qquad \text{for all } v, x \in \mathbb{R}^n, \tag{2}$$

where we have set $V = (V_j^i)_{i=1,\ldots,n; j=1,\ldots d}$ and where λ designates a strictly positive constant.

Remark 1.2. Under the uniform ellipticity condition we have $d \ge n$.

The main result of our investigation is reported in the following theorem.

Theorem 1.3. *Consider the event*

$$A = \{X^H \text{ and } X^K \text{ intersect each other over the interval } [0,1]\}.$$

We have

(1) *if $n > 1/H + 1/K$, then $\mathbb{P}(A) = 0$; and*
(2) *if $n < 1/H + 1/K$, then $\mathbb{P}(A) > 0$.*

The rest of the paper is organized as follows. In SECTION 2, we present some preliminary material on rough path theory and stochastic differential equations driven by fractional Brownian motions. The needed results on fractal properties of solutions to equation (1) is summarized in SECTION 3. Finally, we prove our main result in SECTION 4.

2. PRELIMINARY MATERIAL

For some fixed $H > 1/4$, we consider $(\Omega, \mathcal{F}, \mathbb{P})$ the canonical probability space associated with the fractional Brownian motion (in short fBm) with Hurst parameter H. That is, $\Omega = \mathcal{C}_0([0,1])$ is the Banach space of continuous functions vanishing at zero equipped with the supremum norm, $\mathcal{F}$ is the Borel sigma-algebra and $\mathbb{P}$ is the unique probability measure on Ω such that the canonical process $B = \{B_t = (B_t^1, \ldots, B_t^d),\ t \in [0,1]\}$ is a fractional Brownian motion with Hurst parameter H. In this context, let us recall that B is a d-dimensional centered Gaussian process, whose covariance structure is induced by

$$R\left(t,s\right) := \mathbb{E} B_s^j\, B_t^j = \frac{1}{2}\left(s^{2H} + t^{2H} - |t-s|^{2H}\right), \quad s,t \in [0,1] \text{ and } j = 1, \ldots, d. \tag{3}$$

In particular it can be shown, by a standard application of Kolmogorov's criterion, that B admits a continuous version whose paths are γ-Hölder continuous for any $\gamma < H$.

2.1. **Rough path and SDE driven by fBm.**

In this section, we recall some basic results in rough paths theory and how a fractional Brownian motion is lifted to be a rough path. More details can be found in [9] and [15]. For $N \in \mathbb{N}$, recall that the truncated algebra $T^N(\mathbb{R}^d)$ is defined by

$$T^N(\mathbb{R}^d) = \bigoplus_{m=0}^{N} (\mathbb{R}^d)^{\otimes m},$$

with the convention $(\mathbb{R}^d)^{\otimes 0} = \mathbb{R}$. The set $T^N(\mathbb{R}^d)$ is equipped with a straightforward vector space structure plus a multiplication $\otimes$. Let π_m be the projection on the m-th tensor level. Then $(T^N(\mathbb{R}^d), +, \otimes)$ is an associative algebra with unit element $\mathbf{1} \in (\mathbb{R}^d)^{\otimes 0}$.

For $s < t$ and $m \geq 2$, consider the simplex $\Delta_{st}^m = \{(u_1, \ldots, u_m) \in [s,t]^m;\ u_1 < \cdots < u_m\}$, while the simplices over $[0,1]$ will be denoted by Δ^m. A continuous map $\mathbf{x} : \Delta^2 \to T^N(\mathbb{R}^d)$ is called a multiplicative functional if for $s < u < t$ one has $\mathbf{x}_{s,t} = \mathbf{x}_{s,u} \otimes \mathbf{x}_{u,t}$. An important example arises from considering paths x with finite variation: for $0 < s < t$ we set

$$\mathbf{x}_{s,t}^m = \sum_{1 \leq i_1, \ldots, i_m \leq d} \left(\int_{\Delta_{st}^m} dx^{i_1} \cdots dx^{i_m} \right) e_{i_1} \otimes \cdots \otimes e_{i_m},$$

where $\{e_1, \ldots, e_d\}$ denotes the canonical basis of $\mathbb{R}^d$, and then define the truncated signature of x as

$$S_N(x) : \Delta^2 \to T^N(\mathbb{R}^d), \qquad (s,t) \mapsto S_N(x)_{s,t} := 1 + \sum_{m=1}^{N} \mathbf{x}_{s,t}^m.$$

The function $S_N(x)$ for a smooth function x will be our typical example of multiplicative functional. Let us stress the fact that those elements take values in the strict subset $G^N(\mathbb{R}^d) \subset T^N(\mathbb{R}^d)$, called free nilpotent group of step N, and is equipped with the classical

Carnot-Caratheodory norm which we simply denote by $|\cdot|$. For a path $\mathbf{x} \in \mathcal{C}([0,1], G^N(\mathbb{R}^d))$, the p-variation norm of $\mathbf{x}$ is defined to be

$$\|\mathbf{x}\|_{p-\text{var};[0,1]} = \sup_{\Pi \subset [0,1]} \left(\sum_i |\mathbf{x}_{t_i}^{-1} \otimes \mathbf{x}_{t_{i+1}}|^p \right)^{1/p}$$

where the supremum is taken over all subdivisions Π of $[0,1]$.

With these notions in hand, let us briefly define what we mean by geometric rough path (we refer to [9, 15] for a complete overview): for $p \geq 1$, an element $x : [0,1] \to G^{\lfloor p \rfloor}(\mathbb{R}^d)$ is said to be a geometric rough path if it is the p-var limit of a sequence $S_{\lfloor p \rfloor}(x^m)$, where x^m is a sequence of paths over $[0,1]$ that have bounded variation. In particular, it is an element of the space

$$\mathcal{C}^{p-\text{var};[0,1]}([0,1], G^{\lfloor p \rfloor}(\mathbb{R}^d)) = \{\mathbf{x} \in \mathcal{C}([0,1], G^{\lfloor p \rfloor}(\mathbb{R}^d)) : \|\mathbf{x}\|_{p-\text{var};[0,1]} < \infty\}.$$

The existence of a geometric rough path over a fractional Brownian motion is proved in [7].

Proposition 2.1. *Let B be a fractional Brownian motion with Hurst parameter $H > 1/4$. It admits a lift $\mathbf{B}$ as a geometric rough path of order p for any $p > 1/H$.*

Recall the definition of a geometric rough path. Proposition 2.1 asserts that for $H > 1/4$, almost surely, one can find a sequence $B^m(\omega) \in \mathcal{C}([0,1], \mathbb{R}^n)$ with bounded variation such that $S_{\lfloor p \rfloor}(B^m)(\omega)$ converges to $\mathbf{B}(\omega)$ in $\|\cdot\|_{p-\text{var};[0,1]}-$norm. It then follows from Lyons' continuity theorem that we can send m to infinity in

$$X_t^m(\omega) = x + \int_0^t V_0(X_s^m(\omega))ds + \sum_{i=1}^d \int_0^t V_i(X_s^m(\omega))dB_s^{m,i}(\omega),$$

and both sides converges for all $t \in [0,1]$ and for almost all ω. The limit process X_t is called the solution to equation (1).

3. Fractal and hitting properties of X

Let X be the solution to (1). In this section, we summarize some fractal and hitting properties of X, which will be needed in the proof of our main result. Interested readers are referred to [2] and [13] for more details.

For all Borel sets $E \subset \mathbb{R}^n$, we define $\mathcal{P}(E)$ to be the set of all probability measures with compact support in E. For $\mu \in \mathcal{P}(E)$, we let $\mathcal{E}_\alpha(\mu)$ denote the α-dimensional energy of μ, that is,

$$\mathcal{E}_\alpha(\mu) := \iint \mathrm{K}_\alpha(|x-y|)\, \mu(dx)\, \mu(dy), \tag{4}$$

where K_α denotes the α-dimensional Newtonian kernel, that is,

$$\mathrm{K}_\alpha(r) := \begin{cases} r^{-\alpha} & \text{if } \alpha > 0, \\ \log(N_0/r) & \text{if } \alpha = 0, \\ 1 & \text{if } \alpha < 0, \end{cases} \tag{5}$$

where $N_0 > 0$ is a constant. For all $\alpha \in \mathbb{R}$ and Borel sets $E \subset \mathbb{R}^n$, we then define the α-dimensional capacity of E as

$$\operatorname{Cap}_\alpha(E) := \left[\inf_{\mu \in \mathcal{P}(E)} \mathcal{E}_\alpha(\mu) \right]^{-1}, \tag{6}$$

where by convention we set $1/\infty = 0$.

The theorem below on hitting probabilities of X is proved in [2].

Theorem 3.1. *Let X be the solution to equation (1) driven by a d-dimensional fBm B with Hurst parameter $H > 1/4$. Fix $0 < a < b \leq 1$, $M > 0$, and $\eta > 0$ Then whenever $V_1, \ldots, V_d$ satisfy the uniform ellipticity condition (2), there exists two strictly positive constants c_1, c_2 depending on a, b, H, M, n, η such that for all compact sets $E \subseteq [-M, M]^n$,*

$$c_1 \operatorname{Cap}_{n-\frac{1}{H}}(E) \leq \mathbb{P}\left(X([a,b]) \cap E \neq \varnothing\right) \leq c_2 \operatorname{Cap}_{n-\frac{1}{H}-\eta}(E). \tag{7}$$

Denote by $\dim_{\mathcal{H}}(E)$ the Hausdorff dimension of E. The following theorem is borrowed from [13].

Theorem 3.2. *Let X be the same as in the previous theorem. We have almost surely*

$$\dim_{\mathcal{H}} X([0,1]) = \min\left\{n, \frac{1}{H}\right\}.$$

To close the exposition in this section, we remark that the capacity dimension of E is defined by

$$\dim_{\mathcal{C}}(E) = \sup\{\alpha > 0 : \quad \operatorname{Cap}_\alpha(E) > 0\}.$$

It is know by Frostman's theorem (cf. [11] or [12])

$$\dim_{\mathcal{H}} E = \dim_{\mathcal{C}}(E),$$

for every compact subset E of $\mathbb{R}^n$.

4. Proof of Main result

We prove Theorem 1.3 in this section. Consider the event

$$A = \{X^H \text{ and } X^K \text{ intersect each other over the interval } [0,1]\}.$$

We aim to show that: (1) If $n > 1/H + 1/K$, then $\mathbb{P}(A) = 0$; and (2) if $n < 1/H + 1/K$, then $\mathbb{P}(A) > 0$.

Let $X^H[0,1]$ be the sample path of X^H over the interval $[0,1]$. By Theorem 3.2, there exists $\Omega_1 \subset \Omega$ with $\mathbb{P}(\Omega) = 1$ such that

$$\dim_{\mathcal{H}} X^H[0,1](\omega) = \min\left\{n, \frac{1}{H}\right\}, \quad \text{for all } \omega \in \Omega_1.$$

In the discussion below, we fix an $\omega \in \Omega_1$ and consider the single sample path $X^H[0,1](\omega)$. Clearly, it is a compact set in $\mathbb{R}^n$, as the image of the compact time interval $[0,1]$ under the continuous map $X^H_\cdot(\omega) : [0,1] \to \mathbb{R}^n$. By Frostman's theorem,

$$\dim_{\mathcal{C}} X^H[0,1](\omega) = \dim_{\mathcal{H}} X^H[0,1](\omega) = \min\left\{n, \frac{1}{H}\right\}.$$

Recall that in the above $\dim_{\mathcal{C}} E$ is the capacity dimension of E.

Set
$$E = X^H[0,1](\omega) \subset \mathbb{R}^n.$$
We try to see when X^K has positive probability to hit this particular E. By Theorem 3.1, we have for any $\epsilon > 0$, there exist positive constants c_1 and c_2 such that
$$c_1 \mathrm{Cap}_{n-\frac{1}{K}}(E) \le \mathbb{P}\{X^K \text{ visit } E \text{ during time interval } [a,1]\} \le c_2 \mathrm{Cap}_{n-\frac{1}{K}-\epsilon}(E).$$
Where $a > 0$ is a very small number. By the definition of capacity dimension, the following is obvious.

Case 1: Suppose $n > 1/H$, in which
$$\dim_{\mathcal{C}} E = \frac{1}{H}.$$
We have

(a) If $n-1/K > 1/H$; that is $1/K+1/H < n$, then for all small ϵ (such that $n-1/K-\epsilon > 1/H$),
$$\mathrm{Cap}_{n-\frac{1}{K}-\epsilon} E = 0.$$
Hence
$$\mathbb{P}\{X^K \text{ hits } E \text{ during time interval } [a,1]\} = 0.$$
(b) If $n - 1/K < 1/H$; that is, $1/K + 1/H > n$, then
$$\mathrm{Cap}_{n-\frac{1}{K}} E > 0.$$
Hence
$$\mathbb{P}\{X^K \text{ visit } E \text{ during time interval } [a,1]\} > 0.$$

Case 2: Suppose $n \le 1/H$, in which
$$\dim_{\mathcal{C}} E = n.$$
Obviously, we have $n - 1/K < n$ and therefore
$$\mathrm{Cap}_{n-\frac{1}{K}} E > 0.$$
Hence
$$\mathbb{P}\{X^K \text{ visit } E \text{ during time interval } [a,1]\} > 0.$$
To summarize, we have

(i) If $n > 1/H + 1/K$, then almost surely X^K will not hit E on $[a,1]$;
(ii) If $n < 1/H + 1/K$, the with positive probability X^K will hit E on $[a,1]$.

Finally, for each small $a > 0$ consider the event
$$A_a = \{X^H[0,1] \cap X^K[a,1] \neq \varnothing\}.$$
Let $\mathscr{B}_{X^H}$ be the σ-field generated by X^H. We have
$$\mathbb{P}(A_a) = \mathbb{E}[\mathbb{P}(A_a|\mathscr{B}_{X^H})].$$
Conditioning on $\mathscr{B}_{X^H}$ is equivalent to freezing a sample path of X^H. Since X^K and X^H are independent, the argument above implies that

(i) If $n > 1/H + 1/K$, then $\mathbb{P}(A_a|\mathscr{B}_{X^H}) = 0$ almost surely;
(ii) If $n < 1/H + 1/K$, then $\mathbb{P}(A_a|\mathscr{B}_{X^H}) > 0$ almost surely.

Hence we conclude that: (i) If $n > 1/H+1/K$, we have $\mathbb{P}(A_a) = 0$; and (ii) if $n < 1/H+1/K$, then $\mathbb{P}(A_a) > 0$. The proof is thus completed by observing that $\mathbb{P}(A) = \lim_{a\to 0} \mathbb{P}(A_a)$.

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